with the same cylinders fitted with end plates to simulate the case of infinite length. The cylinders were mounted sufficiently rigid to prevent oscillation. The results are presented in Fig. 2. In accordance with earlier findings the Strouhal number of the infinite cylinder is approximately 0.2, computed from the frequency peak protruding from the otherwise turbulent spectrum. In case of the finite cylinder with aspect ratio 4, a regular motion indicated by a distinct frequency in the spectral distribution with $S \approx 0.45$ was found at $Re_D \approx 2.4 \times 10^5$. Apparently the frequency peak tended to become weaker with decreasing Reynolds number and at $Re_D \approx 5 \times 10^4$ a distinct frequency could no longer be discerned. The spectra obtained with the cylinder of aspect ratio 8 showed a shedding frequency with $S = 0.2$ at subcritical Reynolds numbers with the indication to become stronger with Re_D increasing. The evaluation of Roshko's "universal wake Strouhal number," $S_w = fD_w/U_w$ showed for all measurements $S_w \approx 0.2$.

Naumann[2] found that due to the three dimensionality of the flow the separation line is no longer straight and thus the circulation of a separating vortex is no longer constant, which leads to a break up. This seems to be confirmed by some surface patterns presented by Gould, Raymer, and Ponsford.[3] The surface patterns which were observed during this work, however, exhibit a very straight separation line and at the same time there was no indication of vortex shedding. It appears therefore that some additional explanations may be necessary. For any blunt body part of the flow going over the top of the body is being dragged into the wake. The strength of this downwash obviously decreases with increasing Reynolds number and is restricted to a certain length of the cylinder from top downwards, i.e., its over-all influence decreases with increasing aspect ratio. One may consider the downwash flow as a kind of air curtain acting to some extent like a splitter plate. Depending on its over-all strength it dampens the feedback from both sides of the cylinder and thus the shedding mechanism. Two tests were made to check on this model. In the first case the cylinder with end plate was provided with a slit in the base open to atmosphere, so that by action of the low base pressure a weak jet was blown into the wake. This largely damped the shedding which was clearly observed when the slit was blocked. In the second test the downwash in the near region of the finite cylinder was prevented by a small plate behind the cylinder between half height and top. The presence of this plate accounted for the occurrence of regular shedding. The main observations may be summarized as follows: 1) the pressure distribution around a blunt cylinder, being similar to that of the infinite cylinder has the same Re_D — dependence; 2) the region of highly three-dimensional flow in the wake is restricted to the first 3–4 cylinder diameters; 3) turbulence levels in the near wake reach values of approximate 80% of the local mean velocity; 4) macro scales in the nonperiodic wake are almost an order of magnitude smaller than in the periodic, i.e., shedding case. Their mean values over the cylinder height increase linearly with downstream distance after the first 2 to 3 cylinder diameters; 5) the universal wake Strouhal number appears to be a constant for finite and infinite cylinders alike; and 6) vortex shedding from finite cylinders largely depends on Reynolds number and aspect ratio. Increasing aspect ratio as well as increasing Reynolds number lead to a flow behaviour similar to that of the infinite cylinder. In case of the finite cylinder the downwash is considered the damping mechanism for the shedding feedback.

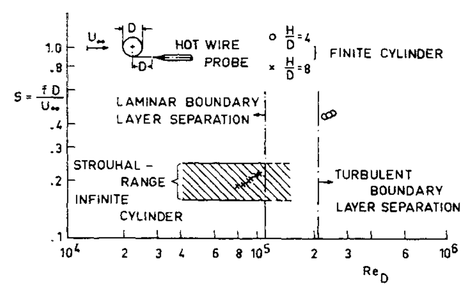

Fig. 1 Strouhal range vs Reynolds number for various cylinders.

References

[1] El Baroudi, M. Y., "Measurements of Two Point Correlations of Velocity near a Circular Cylinder Shedding a Kármán Vortex Street," TN 31, 1961, Univ. of Toronto, Institute of Aerophysics.

[2] Naumann, A. and Quadflieg, H., "Aerodynamic Aspects of Wind Effects on Cylindrical Buildings," *Proceeding of a Symposium on Wind Effects on Building and Structures*, Laughborough Univ., 1968.

[3] Gould, R. W. F., Rymer, W. G., and Ponsford, P. J., "Wind Tunnel Tests on Chimneys of Circular Section at High Reynolds Numbers," *Proceedings of a Symposium on Wind Effects on Building and Structures*, Laughborough Univ., 1968.

Solution of the Equations of Rotational Motion for a Class of Torque-Free Gyrostats

T. R. Kane*

Stanford University, Stanford, Calif.

Introduction

A GYROSTAT is a mechanical system composed of bodies whose relative motions cannot alter the mass geometry of the system; and, if the resultant moment of all external forces about the mass center of the system is equal to zero, the gyrostat is said to be torque free.

This paper deals with torque-free gyrostats G consisting of two bodies, A and B (Fig. 1). The inertia ellipsoid E_A of A for the mass center A^* of A may have three unequal principal diameters, whereas the inertia ellipsoid E_B of B for the mass center B^* of B is presumed to be an ellipsoid of revolution. Furthermore, it is presumed that B is connected to A in such a way that 1) B^* and the axis of revolution of E_B are fixed in A, but B can rotate relative to A about this axis and 2) the inertia ellipsoid E_G of the gyrostat G for the mass center G^* of G is an ellipsoid of revolution whose axis is parallel to that of E_B.

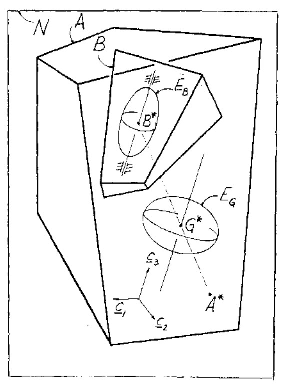

Fig. 1 Schematic representation of a gyrostat.

Received October 20, 1969; revision received February 16, 1970. This work was supported, in part, by NASA, under NGR-05-020-209.

* Professor, Department of Applied Mechanics. Member AIAA.

The solution of the equations of rotational motion for a different class of gyrostats was given by both Masaitis[1] and Leipholz,[2] who fixed B^* on a principal axis of E_A and required that the axis of revolution of E_B pass through A^*. Furthermore, Masaitis assumed that A and B rotate relative to each other "without friction," and Leipholz, when dealing with the special case that arises when E_A is an ellipsoid of revolution, so that the gyrostat under consideration becomes a special member of the class of gyrostats described previously, considered only motions during which B rotates relative to A with constant angular speed, which, as will be shown in the sequel, means that A and B exert no moments on each other about the axis of E_B. Such moments, taken into account in the present analysis, are of interest when one wishes to study effects of energy dissipation on the behavior of gyrostats, a subject of interest in the field of space vehicle attitude control. Thus, what follows may be regarded as an extension of certain portions of the work of Masaitis and Leipholz.

As will be shown, the most general rotational motion of G in a Newtonian reference frame N can be described in remarkably simple terms: B rotates relative to A with an, in general, variable angular speed, and A precesses and spins in a manner similar to that of a single, torque-free, symmetric, rigid body, the only qualitative difference between the behavior of such a body and that of A being that the so-called spin speed is, in general, variable in the case of the gyrostat, whereas it is necessarily constant for the rigid body. What is perhaps surprising is that both the precession speed and the "lean" angle, that is, the angle between the axis of E_G and a certain line fixed in N, remain constant throughout the motion. In any event, these facts make it possible to give a complete description of the state of G at time t as soon as one has formulas that relate the rotational angular velocity $\mathbf{r}$ of B relative to A, the spin angular velocity $\mathbf{s}$, and the precession angular velocity $\mathbf{p}$, on the one hand, to the initial angular speed of B in A and to the initial orientation and angular velocity of A in N, on the other hand. Such formulas will now be developed.

Analysis

In Fig. 1, $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ designate mutually perpendicular unit vectors oriented in such a way that $\mathbf{c}_3$ is parallel to the axes of revolution of E_B and E_G, but $\mathbf{c}_1$ and $\mathbf{c}_2$ are *not* fixed in A or in B. Symbols denoting moments of inertia of interest are defined in terms of these unit vectors and the inertia dyadics $\mathbf{I}^G$ of G for G^* and $\mathbf{I}^B$ of B for B^* by letting

$$I = \mathbf{c}_1 \cdot \mathbf{I}^G \cdot \mathbf{c}_1 = \mathbf{c}_2 \cdot \mathbf{I}^G \cdot \mathbf{c}_2 \tag{1}$$

$$J = \mathbf{c}_3 \cdot \mathbf{I}^G \cdot \mathbf{c}_3 \tag{2}$$

and

$$K = \mathbf{c}_3 \cdot \mathbf{I}^B \cdot \mathbf{c}_3 \tag{3}$$

As B is constrained to rotate relative to A about a line parallel to $\mathbf{c}_3$, the angular velocity $\mathbf{r}$ of B relative to A can be expressed as

$$\mathbf{r} = r\mathbf{c}_3 \tag{4}$$

where r denotes a function of time t. Suppose now that C designates a reference frame in which $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are fixed. Then, as $\mathbf{c}_3$ is parallel to the axis of revolution of E_B, and as this axis is fixed in A, the angular velocity of A relative to C, which will be denoted by $\mathbf{s}$, is necessarily parallel to $\mathbf{c}_3$ and, therefore, can be expressed as

$$\mathbf{s} = s\mathbf{c}_3 \tag{5}$$

where s is a function of t. As will be seen presently, it is the introduction of the reference frame C, that is, the choice of s, that ultimately furnishes the key to the solution of the problem at hand. However, no matter how s is chosen, the angular velocity of C relative to N, to be denoted by $\mathbf{p}$, can always be expressed in terms of functions p_1, p_2, and p_3 of t as

$$\mathbf{p} = p_1\mathbf{c}_1 + p_2\mathbf{c}_2 + p_3\mathbf{c}_3 \tag{6}$$

and the angular velocities, $\mathbf{a}$ and $\mathbf{b}$, of A and B relative to N are then given by†

$$\mathbf{a} = \mathbf{p} + \mathbf{s} \underset{(6,\,5)}{=} p_1\mathbf{c}_1 + p_2\mathbf{c}_2 + (p_3 + s)\mathbf{c}_3 \tag{7}$$

$$\mathbf{b} = \mathbf{a} + \mathbf{r} \underset{(7,\,4)}{=} p_1\mathbf{c}_1 + p_2\mathbf{c}_2 + (p_3 + s + r)\mathbf{c}_3 \tag{8}$$

The angular momentum $\mathbf{H}^G$ of G with respect to G^* in N can be expressed as

$$\mathbf{H}^G = Ip_1\mathbf{c}_1 + Ip_2\mathbf{c}_2 + [J(p_3 + s) + Kr]\mathbf{c}_3 \tag{9}$$

Differentiating $\mathbf{H}^G$ with respect to time in N, and equating the result to zero, one obtains three equations of motion. A fourth equation results from differentiating $\mathbf{c}_3 \cdot \mathbf{H}^B$ with respect to time, where $\mathbf{H}^B$ is the angular moment of B with respect to B^* in N. Keeping in mind that E_B is an ellipsoid of revolution, and equating the time derivative of $\mathbf{H}^B$ in N to the moment $\mathbf{M}$ about B^* of all forces exerted by A on B, one thus finds that the five quantities p_1, p_2, p_3, r, and s must satisfy the four equations

$$I\dot{p}_1 + [(J - I)p_3 + Js + Kr]p_2 = 0 \tag{10}$$

$$I\dot{p}_2 - [(J - I)p_3 + Js + Kr]p_1 = 0 \tag{11}$$

$$J(\dot{p}_3 + \dot{s}) + K\dot{r} = 0 \tag{12}$$

$$(\dot{p}_3 + \dot{s} + \dot{r})K = \mathbf{c}_3 \cdot \mathbf{M} \tag{13}$$

The excess of unknowns over equations is attributable to the fact that s is a function of t which was introduced solely for the purpose of facilitating the analysis and, therefore, may be chosen at will. A choice which, indeed, simplifies the subsequent analysis is

$$s = [(I - J)p_3 - Kr]/J \tag{14}$$

for this permits one to replace Eqs. (10–13) with

$$\dot{p}_1 = 0 \tag{15}$$

$$\dot{p}_2 = 0 \tag{16}$$

$$\dot{p}_3 = 0 \tag{17}$$

$$\dot{r} = [J/K(J - K)]\mathbf{c}_3 \cdot \mathbf{M} \tag{18}$$

[It follows from the definitions of J and K that J exceeds K, so that the denominator in Eq. (18) is positive.]

Integration of Eqs. (15–17) gives

$$p_1 = \bar{p}_1,\; p_2 = \bar{p}_2,\; p_3 = \bar{p}_3 \tag{19}$$

where $\bar{p}_i$ denotes the initial value of p_i, and substitution from Eqs. (14) and (19) into Eq. (9) shows that $\mathbf{p}$ is parallel to $\mathbf{H}^G$. Thus, the angular velocity of C relative to N is seen to be a vector of constant magnitude and fixed direction in N, which means that C performs a simple rotational motion relative to N. This is the aforementioned precession. To express the precession angular velocity $\mathbf{p}$ in terms of initial values of quantities directly associated with the motion of G, one may substitute from Eqs. (14) and (19) into Eq. (7), which gives

$$\mathbf{a} = \bar{p}_1\mathbf{c}_1 + \bar{p}_2\mathbf{c}_2 + [(I\bar{p}_3 - Kr)/J]\mathbf{c}_3 \tag{20}$$

and, defining, $\bar{a}_i$ as

$$\bar{a}_i = \mathbf{a} \cdot \mathbf{c}_i|_{t=0} \tag{21}$$

one then finds that $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ are related to $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, and $\bar{r}$, the initial value of r, as follows:

$$\bar{p}_1 = \bar{a}_1,\; \bar{p}_2 = \bar{a}_2,\; \bar{p}_3 = (J\bar{a}_3 + K\bar{r})/I \tag{22}$$

† Numbers beneath signs of equality are intended to direct attention to corresponding equations.

Equations (6), (19), and (22) now permit one to express **p** as

$$\mathbf{p} = \bar{a}_1\mathbf{c}_1 + \bar{a}_2\mathbf{c}_2 + [(J\bar{a}_3 + K\bar{r})/I]\mathbf{c}_3 \quad (23)$$

By construction, the unit vector $\mathbf{c}_3$ is parallel to a line that is fixed in both A and B. Thus, the angle θ between this line and the vector **p** is given by

$$\theta = \cos^{-1}(\mathbf{c}_3\cdot\mathbf{p}/|\mathbf{p}|) \quad (24)$$

or

$$\theta \underset{(23,24)}{=} \cos^{-1}\frac{J\bar{a}_3 + K\bar{r}}{[I^2(\bar{a}_1{}^2 + \bar{a}_2{}^2) + (J\bar{a}_3 + K\bar{r})^2]^{1/2}} \quad (25)$$

But **p** is parallel to $\mathbf{H}^G$, and $\mathbf{H}^G$ is fixed in N. Consequently, Eq. (25) gives the angle between the axis of revolution of E_B (or of E_G) and a line fixed in N.

Having determined the motion of C relative to N, we next consider the motions of A relative to C, and B relative to A. As regards the former, it is only necessary to recall that **s** denotes the angular velocity of A relative to C and to observe that Eqs. (5, 14, and 22) yield

$$\mathbf{s} = \{[(I - J)/J][(J\bar{a}_3 + K\bar{r})/I] - (K/J)r\}\mathbf{c}_3 \quad (26)$$

from which it is apparent that A spins relative to C with an angular speed that depends on the motion of B relative to A, that is, on r. Thus, the two motions under consideration are intimately related to each other, and an assumption regarding the interaction of A and B must be made before one can arrive at final expressions for **s** and **r**. Three kinds of interaction will be considered: B constrained to remain fixed in A, so that G moves as a rigid body; B completely free to rotate relative to A (Leipholz's case); and B subjected to the action of a torque which retards rotation of B relative to A, the magnitude of the torque being taken proportional to the relative rotation rate r, as might be the case if forces are transmitted from A to B through a viscous medium.

The motion of B relative to A is governed by Eq. (18), which shows that B remains permanently fixed in A if $\mathbf{c}_3\cdot\mathbf{M} = 0$ and $r = 0$ at $t = 0$. In other words, strange as it may seem, no moment about the axis of rotation of B in A is required to prevent B from rotating relative to A. Expressions for **p**, **s**, and **r** suitable for the description of rigid body motions of G are [see Eqs. (23, 26, and 4)]

$$\mathbf{p}_R = \bar{a}_1\mathbf{c}_1 + \bar{a}_2\mathbf{c}_2 + \frac{J}{I}\bar{a}_3\mathbf{c}_3,\ \mathbf{s}_R = \frac{I - J}{I}\bar{a}_3\mathbf{c}_3,\ \mathbf{r}_R = 0 \quad (27)$$

If B is completely free to rotate relative to A, that is, if $\mathbf{c}_3\cdot\mathbf{M}$ is equal to zero, r, in accordance with Eq. (18), must remain constant, and thus equal to its initial value $\bar{r}$. The free rotor forms of **p**, **s**, and **r** are thus,

$$\mathbf{p}_F = \bar{a}_1\mathbf{c}_1 + \bar{a}_2\mathbf{c}_2 + [J\bar{a}_3 + K\bar{r}/I]\,\mathbf{c}_3$$

$$\mathbf{s}_F = \left(\frac{I - J}{I}\bar{a}_3 - \frac{K}{I}\bar{r}\right)\mathbf{c}_3,\ \mathbf{r}_F = \bar{r}\mathbf{c}_3 \quad (28)$$

Finally, when B's rotation relative to A is resisted by a torque whose magnitude is proportional to r, the right-hand member of Eq. (18) can be expressed as $-kr$, where k is a positive constant, and integration of Eq. (18) then yields $r = \bar{r}\exp(-kt)$ so that the expressions for **p**, **s**, and **r** associated with simple viscous damping are

$$\mathbf{p}_V = \bar{a}_1\mathbf{c}_1 + \bar{a}_2\mathbf{c}_2 + [(J\bar{a}_3 + K\bar{r})/I]\mathbf{c}_3 \quad (29)$$

$$\mathbf{s}_V = \left\{\frac{I - J}{I}\bar{a}_3 - \frac{K}{I}\bar{r}\left[1 - \frac{J}{I}\left(1 - e^{-kt}\right)\right]\right\}\mathbf{c}_3$$

$$\mathbf{r}_V = \bar{r}e^{-kt}\mathbf{c}_3 \quad (30)$$

Conclusion

The motion of G in N is completely characterized by the vectors **p**, **s**, and **r**. General expressions for these are given in Eqs. (23, 26, and 4), respectively, where r denotes that solution of Eq. (18) which satisfies the condition $r = \bar{r}$ at $t = 0$. Special forms of the expressions for **p**, **s**, and **r** appear in Eqs. (27–30). Equation (23) shows that the angular velocity of precession is affected by rotation of B relative to A, but that only the initial value of the relative rotation rate, not its entire time history, plays a role. By way of contracts, the spin vector **s** and the relative rotation vector **r** depend intimately on the relative motion of A and B, as may be seen by reference to Eqs. (27, 28, and 30). Finally, Eq. (25) shows that the angle between the angular momentum of G with respect to G^* and the axis of revolution of the ellipsoid E_G remains constant regardless of the nature of the interaction of A and B. Thus, even in the presence of energy dissipation, there is no tendency for a gyrostat of the kind under consideration to "right" itself.

References

[1] Masaitis, C., "On the Motion of Two Linked Bodies," *Archive for Rational Mechanics and Analysis*, Vol. 8, 1961, pp. 23–25.

[2] Leipholz, H., "Ein Beitrag zu dem Problem des Kreisels mit drehzahlabhängiger Selbsterregung," *Ingenieur Archiv*, Vol. 32, No. 4, May 1963, pp. 255–285.

Stiffnesses of an Elastic Filler Constrained between a Rigid Sphere and a Rigid Ellipsoidal Shell

S. S. Chiu*
Sandia Laboratories, Livermore, Calif.

1. Introduction

There exists a structural dynamics problem, related to a flight vehicle, that deals with a spherical mass packaged inside an oblong shell. To predict the dynamic response of the vehicle, the stiffnesses of the filler must be determined. This problem is idealized to the case of a rigid sphere surrounded by an elastic filler which, in turn, is contained within a rotationally symmetric ellipsoidal shell (Fig. 1). The filler is assumed to have a low Poisson's ratio and to consist of nested ellipsoidal layers. Furthermore, the stresses in each layer are assumed to be distributed similarly and their magnitudes to vary inversely as the surface area of the layer. The effects of the spherical inclusion's motion, relative to the shell, on the stiffness of the elastic medium are calculated. A spherical outside shell is also considered as a limiting case of the ellipsoid. The results compare favorably with available analytical data.

2. Formal Statement of Problem

As explained later, the problem requires two sets of spherical coordinates (r,β,ξ) and (r,ϕ,θ) and the Cartesian coordinates (x,y,z). The relationships between the coordinates are

$$x = r\sin\beta\sin\xi = r\sin\phi\sin\theta \quad (1)$$

$$y = r\sin\beta\cos\xi = r\cos\phi \quad (2)$$

$$z = r\cos\beta = r\sin\phi\cos\theta \quad (3)$$

Received June 23, 1969; revision received February 24, 1970. This work was supported by the U. S. Atomic Energy Commission, Contract AT-(29-1)-789.

*Staff member, Analytical Mechanics Division.